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Instanton counting, Macdonald function and the moduli space of D-branes

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Abstract: We argue the connection of Nekrasov's partition function in the Ω background and the moduli space of D-branes, suggested by the idea of geometric engineering and Gopakumar-Vafa invariants. In the instanton expansion of $\mathcal{N} = 2$ SU(2) Yang-Mills theory the Nakrasov's partition function with equivariant parameters ϵ_1, ϵ_2 of toric action on $\mathbb{C}^2$ factorizes correctly as the character of $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ spin representation. We show that up to two instantons the spin contents are consistent with the Lefschetz action on the moduli space of $D2$-branes on (local) $\mathbf{F}_0$. We also present an attempt at constructing a refined topological vertex in terms of the Macdonald function. The refined topological vertex with two parameters of T^2 action allows us to obtain the generating functions of equivariant χ_y and elliptic genera of the Hilbert scheme of n points on $\mathbb{C}^2$ by the method of topological vertex.

Keywords: Solitons Monopoles and Instantons, Topological Field Theories, D-branes.

Contents

1. Introduction

Gravity/gauge theory correspondence is one of important subjects in the recent developments of non-perturbative string theory and D-branes are the crucial dynamical objects for understanding the correspondence. In topological theory the correspondence can be established more precisely as equalities among partition functions of topological string/gauge theory. Since the topological partition function gives the generating function of topological invariants, the equalities imply rather surprising mathematical conjectures on the relation of invariants that seem to have quite different origin at the first sight. The first example was given by the geometric transition based on the duality of the resolved conifold and the deformed conifold [1, 2]. In A-model picture the geometric transition implies the equivalence of the Gromov-Witten invariants of local $\mathbf{P}^1$ (the resolved conifold side) and the Chern-Simons invariants of S^3 (the deformed conifold side), while in B-model picture it leads to the Dijkgraaf-Vafa proposal of matrix model computation of effective superpotential of $\mathcal{N} = 1$ supersymmetric Yang-Mills (SYM) theory [3]. Thanks to a technical tool of topological vertex arising from the idea of geometric transition [4−8], more examples of topological string/gauge theory correspondence have been found in the last years. Among them are the equivalence of the Nekrasov's partition function for instanton counting of four dimensional gauge theory and topological string amplitudes of local toric Calabi-Yau manifold [9−13], and Gromov-Witten/Donaldson-Thomas correspondence [14−17]. Regarding these topological theories as one of the "corners" of a hypothetical topological theory like the picture of M theory for the duality web of perturbative string theories, there have been a few proposal of unifying topological theory [18, 19] that accommodates all the correspondences and encodes the idea of topological S-duality [20, 21].

There is a rather long history of attempts to recover the instanton expansion of the Seiberg-Witten theory from the calculus of multi-instantons (see [22] and references thererin). The main issue was the measure on the instanton moduli space based on the ADHM construction. The topological nature of the instanton amplitudes was understood from the existence of a BRST-type operator [23] and consequently the localization theorem was successfully applied [24]. In the end Nekrasov correctly computed the partition function that produced the instanton expansion of the Seiberg-Witten prepotential by putting $\mathcal{N} = 2$ SYM in the Ω background [25]. The Ω background has a natural interpretation from the six dimensional viewpoint and it gives a coupling with gravity in a special form. The background has two parameters ϵ_1 and ϵ_2, which define a twisting of two complex coordinates (z_1, z_2) of the flat (euclidean) space $\mathbb{R}^4 \simeq \mathbb{C}^2$;

$$(z_1, z_2) \mapsto (e^{i\epsilon_1} z_1, e^{i\epsilon_2} z_2)\,. \tag{1.1}$$

Mathematically they are equivariant parameters corresponding to the weights of $T = \mathbb{C}^\times \times \mathbb{C}^\times$ action on $\mathbb{C}^2$. For any manifold X with T action the equivariant cohomology ring $H_T^*(X, \mathbb{Q})$ is a module over $H_T^*(\mathrm{pt}) \simeq \mathbb{Q}[t, q]$, $t := e^{\epsilon_1}, q := e^{-\epsilon_2}$. When we consider the special case $\hbar := \epsilon_1 = -\epsilon_2$, the Nekrasov's partition function has the following expansion

$$Z_{\mathrm{Nek}} = \exp\left(\sum_{g\geq 0} \hbar^{2g-2} F_g\right)\,, \tag{1.2}$$

and the lowest term F_0 gives the instanton part of the Seiberg-Witten prepotential [26−28]. The fact that Z_{Nek} can be expanded like the genus expansion in string theory is not an accident and it turns out Z_{Nek} is exactly reproduced as a (closed) topological string amplitudes on an appropriate background. The amplitudes of our concern are those on local toric Calabi-Yau manifolds and the computation is facilitated by making use of topological vertex introduced in [7]. The topological vertex is related to the Hopf link invariants of the Chern-Simons theory in large-N limit and they are given by (the special values of) the Schur function [29]. In the theory of symmetric functions, a generalization of the Schur function with two parameters is known and it is called Macdonald function [30]. Then it is natural to expect that the Nekrasov's partition function in general Ω background is expressed in terms of the Macdonald function. The aim of this article is to explore the relation of the Macdonald function to the Nakrasov's partition function.

The counting of BPS states in five dimensional theories is one of natural ways to understand the equivalence of the Nekrasov's partition function for instanton counting in four dimensions and topological string amplitudes which are given by Gopakumar-Vafa

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invariants [31, 32]. In the space-time interpretation of topological string amplitudes we consider a constant self-dual graviphoton background and this is in fact what Nekrasov did in his Ω background with the constraint $\epsilon_1 + \epsilon_2 = 0$. In section two we consider Nekrasov's formula for the SU(2) gauge theory and check that, even if $\epsilon_1 + \epsilon_2 \neq 0$ and the self-duality is lost, it factorizes correctly as the character of $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ spin representation of five dimensional massive particles. Part of similar results was already reported in [33] and we will extend it up to two instantons. The idea of geometric engineering tells us that SU(2) gauge theory can be geometrically engineered by type-IIA compactification on local Calabi Yau space $K_{\mathbf{F}_0}$; the canonical bundle of the Hirzebruch surface $\mathbf{F}_0 \simeq \mathbf{P}^1 \times \mathbf{P}^1$. In the attempts at a mathematical definition of the Gopakumar-Vafa invariants, it has been argued that the spin contents of five dimensional theory are identified with the Lefschetz decomposition of the cohomology ring of the moduli space of $D2$-branes [34, 35]. We find that the spin contents obtained by Nekrasov's formula are consistent with this interpretation of Gopakumar-Vafa invariants on local $\mathbf{F}_0$.

The fact that Nekrasov's partition function in general Ω background allows a factorization in terms of the spectrum of five dimensional massive particles strongly suggests the existence of topological vertex with two parameters. In section three we make a modest step towards constructing such a refined topological vertex. We note that a similar refinement in the context of open string theory and its relation to knot invariants have been discussed quite recently [36]. We propose the refined topological vertex in terms of the specialization of the Macdonald function that is a two parameter generalization of the Schur function. As a preliminary example we consider U(1) gauge theory with a massive adjoint hypermultiplet, which is mathematically related to the Hilbert scheme $\mathrm{Hilb}^n\,\mathbb{C}^2$ of n points on $\mathbb{C}^2$. We show that the generating functions of equivariant χ_y genera and elliptic genera of $\mathrm{Hilb}^n\,\mathbb{C}^2$ are obtained by the method of topological vertex. Several technical points on partitions and Macdonald functions are summarized in appendices.

We have checked that Nekrasov's formula for $\mathrm{SU}(N)$ gauge theory in general Ω background is reproduced by the refined topological vertex proposed in this paper. However, we have the issue of cyclic symmetry of the vertex. Therefore, the amplitude obtained by the method of topological vertex may not be unique for a given toric diagram. The results will be reported elsewhere [37].

2. Nekrasov's partition function and the degeneracy of BPS states

The Ω-background of Nekrasov has two parameters (ϵ_1, ϵ_2) and introduces physically a constant graviphoton field of $\mathcal{N} = 2$ supergravity in four dimensions.[1] The condition $\epsilon_1 + \epsilon_2 = 0$ implies that the graviphoton background is self-dual and thus satisfies the BPS condition. A constant self-dual graviphoton background plays quite a significant role in the space-time interpretation of topological string amplitudes and it is natural that we can relate the Nekrasov's partition function to topological string amplitudes, when $\epsilon_1 + \epsilon_2 = 0$. It gives an A-model amplitude that only depends on the Kähler parameters and is invariant under the deformation of complex structure of the target.

[1] In fact this point becomes more clear, if the theory is lifted to five dimensions [38].

In a general Ω-background, where we have no longer $\epsilon_1+\epsilon_2=0$, the background breaks the BPS condition. But due to their topological nature, we expect that the instanton amplitudes in four dimensional gauge theory are unaffected by the deformations. From the viewpoint of topological A-model, one may worry about the fact that the independence of the amplitudes under the deformation is not protected by supersymmetry. However, as has been pointed out in [33], when we consider non-compact Calabi-Yau manifolds,[2] there are no (well-defined) deformations of complex structure and we do not have to mind the jump of the spectrum of the BPS states. In other words it is not required to take a supersymmetric trace with respect to the spin contents of $\mathrm{SU}(2)_R$ action.

2.1 Moduli space of $D2$-branes

Let us review the relation of (generalized) Gopakumar-Vafa invariants[3] and the Lefschetz action on the cohomology of the moduli space $\widetilde{\mathcal{M}}_\beta$ of $D2$-branes wrapping on the holomorphic two-cycle $\beta \in H_2(X,\mathbb{Z})$. The cohomology classes in $H^*(\widetilde{\mathcal{M}}_\beta)$ are in one to one correspondence with the BPS states in a five dimensional theory obtained by a compactification of M-theory on a Calabi-Yau 3-fold X. These BPS states are labeled by the representation of the spacial rotation group $\mathrm{SO}(4) \simeq \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ in five dimensions and the degeneracy of the BPS states can be written as

$$\left[\left(\frac{1}{2},0\right) \oplus 2(0,0)\right] \otimes \bigoplus_{(j_L,j_R)} N_\beta^{(j_L,j_R)} \left[(j_L,j_R)\right]. \tag{2.1}$$

Since the BPS state preserves half the supersymmetry, we always have the structure of the half hypermultiplet $\left[\left(\frac{1}{2},0\right) \oplus 2(0,0)\right]$. The integers $N_\beta^{(j_L,j_R)}$ denote the number of BPS states with the central charge of the homology class β and $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ spin (j_L,j_R). The moduli space of $D2$-branes consists of the deformation of the holomorphic cycle in X together with the moduli of flat (=stable) U(1) bundle over it. Thus we have a fibration $\pi : \widetilde{\mathcal{M}}_\beta \to \mathcal{M}_\beta$, where the base $\mathcal{M}_\beta$ is the moduli space of the two-cycle β without a choice of flat bundle. If the two-cycle is generically the Riemann surface of genus g, then the generic fiber is T^{2g}, the jacobian variety of the Riemann surface. Both $\widetilde{\mathcal{M}}_\beta$ and $\mathcal{M}_\beta$ are Kähler manifolds and we have the Lefschetz action on the cohomology group defined by the multiplication of a Kähler form. It has been argued that the $\mathrm{SU}(2)_L$ spin is identified with the Lefschetz decomposition along the fiber direction (a relative Lefschetz action) and the $\mathrm{SU}(2)_R$ corresponds to the action on the base space [34, 35]. Thus we have the following decomposition of the cohomology of the moduli space of $D2$ branes;

$$H^*(\widetilde{\mathcal{M}}_\beta) = \sum N_\beta^{(j_1,j_2)} \left[\left(j_1^{\mathrm{fiber}}, j_2^{\mathrm{base}}\right)\right]. \tag{2.2}$$

In particular this identification implies that the $\mathrm{SU}(2)_R$ spin contents with the highest $\mathrm{SU}(2)_L$ spin are given by the Lefschetz decomposition of the cohomology of the base space $\mathcal{M}_\beta$.

[2] This is the case in the geometric engineering of four dimensional gauge thoery.

[3] For recent developments in Gopakumar-Vafa invariants, see [39 – 41].

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We thus expect the Nekrasov's partition function in a general Ω background should be related to the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ spin contents of BPS states obtained by wrapping $M2$ branes on a supersymmetric(=holomorphic) two-cycle. It is a very hard mathematical problem to work out such a Lefschetz decomposition of $H^*(\widetilde{\mathcal{M}}_\beta)$ in general. Fortunately we have some information on this issue for local $\mathbf{F}_0 = \mathbf{P}^1 \times \mathbf{P}^1$, to which the SU(2) pure Yang-Mills theory is expected to correspond. In the following we will obtain a few predictions on the Lefschetz decomposition from the Nekrasov's partition function for SU(2) Yang-Mills theory and show that they are consistent with the known results on the moduli space of $D2$ branes on local $\mathbf{F}_0$.

In the space-time interpretation of topological string amplitudes the free energy $F = \log Z = \sum_{g=0}^\infty g_s^{2g-2} F_g$ is obtained by summing over the contribution of the BPS particle spectrum with multiplicities $N_\beta^{(j_L,j_R)}$ to the low energy effective action. We thus obtain in general

$$F(q,t;Q_\beta) = \sum_{\beta\in H_2(X,\mathbb{Z})} \sum_{n=1}^\infty \sum_{j_L,j_R} \frac{N_\beta^{(j_L,j_R)}}{n(q^{n/2}-q^{-n/2})(t^{n/2}-t^{-n/2})} \times \\ \times \left((qt)^{-n\cdot j_L} + \cdots + (qt)^{n\cdot j_L}\right)\left((q/t)^{-n\cdot j_R} + \cdots + (q/t)^{n\cdot j_R}\right) Q_\beta^n, \quad (2.3)$$

where we have introduced a constant graviphoton background with $F_{12} = \epsilon_1, F_{34} = \epsilon_2$ and put $t := e^{\epsilon_1}, q := e^{-\epsilon_2}$. The first summation is over the second homology classes β of the target space X with the integer coefficients. In the BPS state counting the two cycle β should be supersymmetric, that means it is holomorphic. Hence only non-negative part contributes to the sum over $H_2(X,\mathbb{Z})$. $Q_\beta = e^{-t_\beta}$ and $t_\beta = \int_\beta \omega$, with ω being a Kähler form of the target space, namely t_β is the Kähler parameter associated with the two cycle β. The summation over n accounts for the multi-covering. Since the multiplicity $N_\beta^{(j_L,j_R)}$ is independent of n, we can compute it by extracting the $n=1$ term from the sum of the multi-covering. It is convenient to make a change of variables; $q = uv, t = u/v$. Then the free energy is

$$F(q,t;Q_\beta) = \sum_{\beta\in H_2(X,\mathbb{Z})} \sum_{j_L,j_R} \sum_{n=1}^\infty \frac{N_\beta^{(j_L,j_R)} u^n v^n}{n(u^n v^n - 1)(u^n - v^n)} \times \\ \times \left(\frac{u^{2n\cdot j_L+1} - u^{-2n\cdot j_L-1}}{u-u^{-1}}\right) \cdot \left(\frac{v^{2n\cdot j_R+1} - v^{-2n\cdot j_R-1}}{v-v^{-1}}\right) Q_\beta^n. \quad (2.4)$$

Note that

$$\chi_n(q) := q^n + q^{n-2} + \cdots + q^{-n+2} + q^{-n} = \frac{q^{n+1}-q^{-n-1}}{q-q^{-1}}, \quad (2.5)$$

is the character of the irreducible representation of SU(2) with spin $n/2$.

2.2 $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ decomposition

In [42, 43, 27] the Nekrasov's partition function for $\mathrm{SU}(N_c)$ Yang-Mills theory in a general Ω background is written in the following form;

$$Z(\epsilon_1,\epsilon_2,\vec{a},\Lambda) = \sum_{\vec{Y}} \frac{\Lambda^{|\vec{Y}|}}{\prod_{\alpha,\beta} n_{\alpha,\beta}^{\vec{Y}}(\epsilon_1,\epsilon_2,\vec{a})}, \quad (2.6)$$

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where

$$n_{\alpha,\beta}^{\vec{Y}} = \prod_{s\in Y_\alpha} (-\ell_{Y_\beta}(s)\epsilon_1 + (a_{Y_\alpha}(s)+1)\epsilon_2 + a_\beta - a_\alpha) \prod_{t\in Y_\beta} ((\ell_{Y_\alpha}(t)+1)\epsilon_1 - a_{Y_\beta}(t)\epsilon_2 + a_\beta - a_\alpha) \,. \tag{2.7}$$

We follow the notations in [27]. The right hand side of (2.6) is a summation over N_c-tuples of Young diagrams $\vec{Y} = (Y_1, Y_2, \cdots, Y_{N_c})$ or colored partitions.[4] The total number of boxes of $\vec{Y}$ is denoted by $|\vec{Y}|$ and identified with the instanton number. $\ell_Y(s)$ and $a_Y(s)$ are the leg-length and the arm-length of the Young diagram (see appendix A.1). In deriving (2.6) an equivariant integration over the (framed) instanton moduli space $\mathcal{M}_{ADHM}$ has to be computed and the localization principle is employed. The N_c-tuples of Young diagrams $\vec{Y}$ are in one to one correspondence with the fixed points of the toric action on $\mathcal{M}_{ADHM}$. The denominator $\prod_{\alpha,\beta} n_{\alpha,\beta}^{\vec{Y}}$ is the product of the eigenvalues (weights) of the toric action at the fixed point $\vec{Y}$. Note that the number of the factors in $\prod_{\alpha,\beta} n_{\alpha,\beta}^{\vec{Y}}$ is $2kN_c$ and this is nothing but the (complex) dimensions of $\mathcal{M}_{ADHM}$.

Introducing the notations $t := e^{\epsilon_1}, q := e^{-\epsilon_2}$ and $Q_{\beta\alpha} := e^{a_\beta - a_\alpha}$, we consider the following five dimensional lift;

$$N_{\alpha,\beta}^{\vec{Y}} = \prod_{s\in Y_\alpha} \left(1 - t^{-\ell_{Y_\beta}(s)} q^{-(a_{Y_\alpha}(s)+1)} Q_{\beta\alpha}\right) \prod_{t\in Y_\beta} \left(1 - t^{\ell_{Y_\alpha}(t)+1} q^{a_{Y_\beta}(t)} Q_{\beta\alpha}\right). \tag{2.8}$$

More explicitly for SU(2) case ($\vec{Y} = (Y_1, Y_2)$ and $Q := Q_{12} = Q_{21}^{-1}$); we have

$$\begin{aligned}
\prod_{\alpha,\beta} N_{\alpha,\beta}^{(Y_1,Y_2)} &= N_{11}^{(Y_1,Y_2)} N_{22}^{(Y_1,Y_2)} N_{12}^{(Y_1,Y_2)} N_{21}^{(Y_1,Y_2)} \\
&= \prod_{s\in Y_1} \left(1 - t^{-\ell_{Y_1}(s)} q^{-(a_{Y_1}(s)+1)}\right)\left(1 - t^{\ell_{Y_1}(s)+1} q^{a_{Y_1}(s)}\right) \times \\
&\quad \times \prod_{t\in Y_2} \left(1 - t^{-\ell_{Y_2}(t)} q^{-(a_{Y_2}(t)+1)}\right)\left(1 - t^{\ell_{Y_2}(t)+1} q^{a_{Y_2}(t)}\right) \times \\
&\quad \times \prod_{s\in Y_1} \left(1 - t^{-\ell_{Y_2}(s)} q^{-(a_{Y_1}(s)+1)} Q^{-1}\right) \prod_{t\in Y_2} \left(1 - t^{\ell_{Y_1}(t)+1} q^{a_{Y_2}(t)} Q^{-1}\right) \times \\
&\quad \times \prod_{s\in Y_2} \left(1 - t^{-\ell_{Y_1}(s)} q^{-(a_{Y_2}(s)+1)} Q\right) \prod_{t\in Y_1} \left(1 - t^{\ell_{Y_2}(t)+1} q^{a_{Y_1}(t)} Q\right).
\end{aligned} \tag{2.9}$$

We note the following symmetry;

$$\prod_{\alpha,\beta} N_{\alpha,\beta}^{(Y_1,Y_2)}(q,t,Q) = \prod_{\alpha,\beta} N_{\alpha,\beta}^{(Y_1^t,Y_2^t)}(t^{-1},q^{-1},Q)\,, \tag{2.10}$$

$$\prod_{\alpha,\beta} N_{\alpha,\beta}^{(Y_1,Y_2)}(q,t,Q) = \prod_{\alpha,\beta} N_{\alpha,\beta}^{(Y_2,Y_1)}(q,t,Q^{-1})\,, \tag{2.11}$$

where Y^t means the transpose of the Young diagram. Since these appear in pairs in the sum with a fixed instanton number, the final expression should be symmetric under $(q,t) \leftrightarrow (t^{-1},q^{-1})$ and $Q \leftrightarrow Q^{-1}$.

[4] In this paper we often identify Young diagrams, partitions and representations of general linear group.

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Let us look at examples at lower instanton numbers. At one instanton we have either $\overrightarrow{Y} = (\square, \bullet)$ or $\overrightarrow{Y} = (\bullet, \square)$ and the partition function is

$$\begin{aligned} Z^{\text{one-inst}}(q,t,Q) = & \frac{1}{(1-q^{-1})(1-t)(1-tq^{-1}Q^{-1})(1-Q)} + \\ & + \frac{1}{(1-q^{-1})(1-t)(1-Q^{-1})(1-tq^{-1}Q)} \,. \end{aligned} \tag{2.12}$$

There are five possibilities of (Y_1, Y_2) for two instanton and we obtain;

$$\begin{aligned} Z^{\text{two-inst}}(q,t,Q) = & \frac{1}{(1-t^{-1}q^{-1})(1-q^{-1})(1-t^2)(1-t)} \times \\ & \times \left[\frac{1}{(1-tq^{-1}Q^{-1})(1-t^2q^{-1}Q^{-1})(1-Q)(1-t^{-1}Q)} + \right. \\ & \left. + \frac{1}{(1-tq^{-1}Q)(1-t^2q^{-1}Q)(1-Q^{-1})(1-t^{-1}Q^{-1})} \right] + \\ & + \frac{1}{(1-q^{-2})(1-q^{-1})(1-tq)(1-t)} \times \\ & \times \left[\frac{1}{(1-tq^{-2}Q^{-1})(1-tq^{-1}Q^{-1})(1-qQ)(1-Q)} + \right. \\ & \left. + \frac{1}{(1-tq^{-2}Q)(1-tq^{-1}Q)(1-qQ^{-1})(1-Q^{-1})} \right] + \\ & + \frac{1}{(1-q^{-1})^2(1-t)^2} \frac{1}{(1-q^{-1}Q^{-1})(1-tQ^{-1})(1-q^{-1}Q)(1-tQ)} \,. \end{aligned} \tag{2.13}$$

It is convenient to use the variable $v^2 := q/t$ and rewrite the one-instanton part of the partition function as follows;

$$(q^{1/2} - q^{-1/2})(t^{1/2} - t^{-1/2}) Z^{\text{one-inst}}(q,t,Q) = \frac{vQ}{1-Q} \left(\frac{v^2}{1-v^2Q} + \frac{1}{1-v^{-2}Q} \right) \,. \tag{2.14}$$

We define the prepotential by

$$F := \log Z = \log(1 + \Lambda \cdot Z^{\text{one-inst}} + \Lambda^2 \cdot Z^{\text{two-inst}} + \cdots) \,. \tag{2.15}$$

Then we have the following expansion of the one instanton contribution to the prepotential $F^{\text{one-inst}} = Z^{\text{one-inst}}$;

$$\begin{aligned} F^{\text{one-inst}}(q,t,Q) &= \frac{v^2 Q}{(q^{1/2} - q^{-1/2})(t^{1/2} - t^{-1/2})} \left(\sum_{n=0}^{\infty} Q^n \right) \left(\sum_{k=0}^{\infty} (v^{2k+1} + v^{-2k-1}) Q^k \right) \\ &= \frac{v^2 Q}{(q^{1/2} - q^{-1/2})(t^{1/2} - t^{-1/2})} \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} (v^{2k+1} + v^{-2k-1}) \right) Q^n \,. \end{aligned} \tag{2.16}$$

Comparing this expansion with the general form (2.4) of the free energy, we find

$$N_{B+nF}^{(j_L, j_R)} = \delta_{j_L,0} \delta_{j_R, n+\frac{1}{2}} \,, \tag{2.17}$$

and that one instanton part is saturated by the $j_L = 0$ contributions. As has been argued in [33], this result is consistent with the geometry of the moduli space of $D2$ branes. It is known that a curve of bi-degree (a, b) in $\mathbf{P}^1 \times \mathbf{P}^1$ has total degree $d = a + b$ and generically genus $g = (a-1)(b-1)$. Furthermore the moduli space of such curves (without the flat bundle over them) is shown to be $\mathbf{P}^{(a+1)(b+1)-1}$. When $a = 1$ the curve has generically genus zero and the fiber is trivial, implying there is no non-vanishing left spin. The moduli space of the curve is $\mathbf{P}^{2n+1}$ and the Lefschetz decomposition of $H^*(\mathbf{P}^{2n+1})$ gives a single multiplet of spin $n + 1/2$.

The prepotential of two instanton part is given by

$$F^{\text{two-inst}}(q,t,Q) = Z^{\text{two-inst}}(q,t,Q) - \frac{1}{2}(Z^{\text{one-inst}}(q,t,Q))^2 \,, \tag{2.18}$$

and the effect of multicovering is subtracted by

$$\widetilde{F}^{\text{two-inst}}(q,t,Q) = F^{\text{two-inst}}(q,t,Q) - \frac{1}{2}Z^{\text{one-inst}}(q^2,t^2,Q^2) \,. \tag{2.19}$$

From (2.12) and (2.13) we obtain

$$\widetilde{F}^{\text{two-inst}}(u,v,Q) = \frac{uv}{(uv-1)(u-v)} \frac{N(u,v,Q)}{D(u,v,Q)} \cdot (v^2 Q)^2 \,, \tag{2.20}$$

where

$$\begin{aligned} D(u,v,Q) = &\left(1 - v^2 Q\right)\left(1 - v^4 Q^2\right)\left(1 - uv^3 Q\right)\left(1 - u^{-1}v^3 Q\right) \times \\ &\times \left(1 - v^{-2}Q\right)\left(1 - v^{-4}Q^2\right)\left(1 - uv^{-3}Q\right)\left(1 - u^{-1}v^{-3}Q\right) , \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} N(q,t,Q) = Q^3 \Big[&\left(v^5 + v^3 + v + v^{-1} + v^{-3} + v^{-5}\right)\left(Q^2 + Q^{-2}\right) + \\ &+ \left(v^5 + v^{-5} - (v^2 + 1 + v^{-2})(u + u^{-1})\right)\left(Q + Q^{-1}\right) + \\ &+ \left(v^5 - v - v^{-1} + v^{-5} - (v^2 + v^{-2})(u + u^{-1})\right) \Big] . \end{aligned} \tag{2.22}$$

The expansion of $\widetilde{F}^{\text{two-inst}}$ with respect to the parameter Q gives

$$\widetilde{F}^{\text{two-inst}}(u,v,Q) = \frac{uv^5 Q^2}{(uv-1)(u-v)} \frac{1}{(u-u^{-1})(v-v^{-1})} \sum_{k=1}^{\infty} GV_{2,k} Q^k \,, \tag{2.23}$$

where

$$GV_{2,k} = \sum_{\ell=1}^{k} U_\ell \sum_{m=1}^{k-\ell+1} \left[\frac{m+1}{2}\right] V_{3\ell+2m+1} \,, \tag{2.24}$$

with $U_n = u^n - u^{-n}$ and $V_n = v^n - v^{-n}$. $[x]$ stands for the integer part of x. Thus we find the following $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ spin contents of BPS particle arising from the homology class $2B + kF$;

$$\bigoplus_{(j_L,j_R)} N_{2B+kF}^{(j_L,j_R)} (j_L, j_R) = \bigoplus_{\ell=1}^{k} \bigoplus_{m=1}^{k-\ell+1} \left[\frac{m+1}{2}\right] \left(\frac{\ell-1}{2}, \frac{3\ell+2m}{2}\right) . \tag{2.25}$$

For lower values of the winding number k, this formula implies

$$
\begin{aligned}
k=1: &\left(0,\frac{5}{2}\right)\\
k=2: &\left(\frac{1}{2},4\right)\oplus\left(0,\frac{7}{2}\right)\oplus\left(0,\frac{5}{2}\right)\\
k=3: &\left(1,\frac{11}{2}\right)\oplus\left(\frac{1}{2},5\right)\oplus\left(\frac{1}{2},4\right)\oplus 2\left(0,\frac{9}{2}\right)\oplus\left(0,\frac{7}{2}\right)\oplus\left(0,\frac{5}{2}\right)\\
k=4: &\left(\frac{3}{2},7\right)\oplus\left(1,\frac{13}{2}\right)\oplus\left(1,\frac{11}{2}\right)\oplus 2\left(\frac{1}{2},6\right)\oplus\left(\frac{1}{2},5\right)\oplus\left(\frac{1}{2},4\right)\oplus 2\left(0,\frac{11}{2}\right)\oplus\\
&\oplus 2\left(0,\frac{9}{2}\right)\oplus\left(0,\frac{7}{2}\right)\oplus\left(0,\frac{5}{2}\right).
\end{aligned}
\tag{2.26}
$$

From the formula (2.25) we find the spin content with the highest left spin is $((k-1)/2,(3k+2)/2)$ for degree k. Again this is consistent with the moduli space of $D2$ branes. The genus of the curves with bi-degree $(2,k)$ is generically $k-1$ and the moduli space is $\mathbf{P}^{3k+2}$. Thus the generic fiber is T^{2k-2} and the highest left spin is $(k-1)/2$. The right spin contents with this highest left spin agrees with the result from the fact the it is identified with the Lefschetz decomposition of $\mathcal{M}_{2B+kF}=\mathbf{P}^{3k+2}$. The subleading terms in the spin decomposition are supposed to come from the degeneration of two cycles to the curve of lower genus. It is an interesting challenge in mathematics to clarify it together with the description of the moduli space of flat line bundles over degenerate curves. At three instanton level the spin contents will get more complicated. We have checked that the leading spin content for the class $3B+kF$ is $(0,7/2),(1,11/2),(2,15/2),\ldots$, which agree to those expected from the geometry of D-brane moduli space.

3. Equivariant genus of $\mathrm{Hilb}^n\ \mathbb{C}^2$: rank one case

In the last section we have seen that the Nekrasov's partition function for SU(2) gauge theory carries the information of the refined BPS state counting in the topological string amplitude on local $\mathbf{P}^1\times\mathbf{P}^1$. It strongly suggests the existence of two parameter generalization of topological vertex. In this section we consider U(1) gauge theory with a massive adjoint hypermultiplet and make an attempt at constructing such a two parameter generalization in terms of the Macdonald function, which is a two parameter generalization of the Schur function. If the Nekrasov's partition function is specialized to the rank one case, it gives the generating function of the character of the equivariant cohomology of the Hilbert scheme $\mathrm{Hilb}^n\ \mathbb{C}^2$ of n points on $\mathbb{C}^2$. In [33] it has been argued that five and six dimensional lifts of the abelian gauge theory with a massive adjoint matter compute the χ_y genus and the elliptic genus of $\mathrm{Hilb}^n\ \mathbb{C}^2$, respectively and that a diagrammatic computation in terms of topological vertex is presented. Let us briefly review their method in five dimensional theory.

3.1 χ_y genus — five dimensional theory

Based on the web-diagram of figure 1, we can write down the partition function in terms of topological vertex $C_{R_1R_2R_3}(q)$ that has a parameter $q=e^{-ig_s}$ (g_s is the string coupling

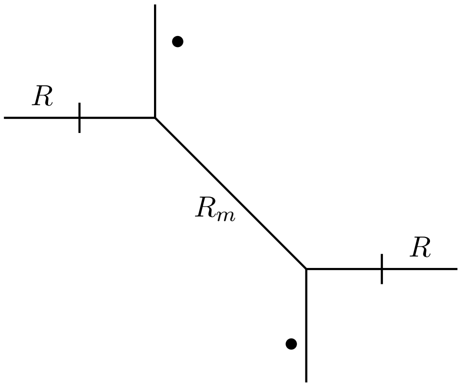

Figure 1: Geometric engineering of five dimensional U(1) theory with adjoint hypermultiplet. Horizontal external lines are identified to make a $D5$-brane wrapping on a circle, The (vertical) distance of the horizontal lines is identified with the mass of the adjoint matter.

constant). It is given by

$$Z(T, T_m; q) = \sum_R e^{-T\cdot|\mu^R|}(-1)^{|\mu^R|} Z_R(T_m; q)\,, \tag{3.1}$$

$$Z_R(T_m; q) = \sum_{R_m} e^{-T_m\cdot|\mu^{R_m}|}(-1)^{|\mu^{R_m}|} C_{\bullet R_m^t R}(q) C_{\bullet R_m R^t}(q)\,, \tag{3.2}$$

where the two Kähler parameters T and T_m are related to the coupling constant τ of the gauge theory and the mass of the adjoint hypermultipet as follows;

$$Q_\tau := e^{2\pi i\tau} = e^{-T-T_m}\,, \qquad Q_m := e^{rm} = e^{-T_m}\,. \tag{3.3}$$

The parameter r is the radius of S^1 of the fifth dimension. We have identified the representations R and R_m assigned to the (internal) edges with the partitions μ^R and μ^{R_m}, respectively. $|\mu^R|$ and $|\mu^{R_m}|$ denote the number of boxes of the corresponding Young diagrams.

Using the expression of the topological vertex $C_{R_1R_2R_3}(q)$ in terms of the Schur function $s_R(x)$ and the Cauchy formula;

$$\sum_R s_R(x) s_{R^t}(y) = \prod_{i,j}(1 + x_i y_j)\,, \tag{3.4}$$

we obtain

$$\begin{aligned} Z_R(T_m, q) &= s_R(q^\rho) s_{R^t}(q^\rho) \prod_{i,j}(1 - Q_m q^{\mu_i^R + \mu_j^{R^t} - i - j + 1}) \\ &= s_R(q^\rho) s_{R^t}(q^\rho) \prod_{k\geq 1}(1 - Q_m q^k)^k \prod_{(i,j)\in\mu^R}(1 - Q_m q^{h(i,j)})(1 - Q_m q^{-h(i,j)})\,, \end{aligned} \tag{3.5}$$

where $h(i,j)$ is the hook length at the box (i,j) in the Young diagram of μ^R. Substituting the specialization of the Schur function;

$$s_R(q^\rho)s_{R^t}(q^\rho) = \frac{q^{\sum_{(i,j)\in\mu^R} h(i,j)}}{\prod_{(i,j)\in\mu^R}(1-q^{h(i,j)})^2}\,, \tag{3.6}$$

we finally find

$$Z = \prod_{k\geq 1}(1-Q_m q^k)^k \sum_R Q^{|\mu^R|} \prod_{(i,j)\in\mu^R} \frac{(1-Q_m q^{h(i,j)})(1-Q_m q^{-h(i,j)})}{(1-q^{h(i,j)})(1-q^{-h(i,j)})} \tag{3.7}$$

$$= \prod_{k\geq 1}(1-Q_m q^k)^k \sum_R Q_\tau^{|\mu^R|} \prod_{(i,j)\in\mu^R} \frac{(1-Q_m q^{h(i,j)})(1-Q_m^{-1} q^{h(i,j)})}{(1-q^{h(i,j)})^2}\,, \tag{3.8}$$

with $Q_\tau = QQ_m$.

The generating function of the equivariant χ_y genera of $\mathrm{Hilb}^n\ \mathbb{C}^2$ is given by [44];

$$\sum_{n=0}^\infty Q^n \chi_y\left(\mathrm{Hilb}^n\ \mathbb{C}^2\right)(t,q) = \sum_\mu Q^{|\mu|}\prod_{s\in\mu}\frac{(1-y\ t^{-\ell(s)}q^{-a(s)-1})(1-y\ t^{\ell(s)+1}q^{a(s)})}{(1-t^{-\ell(s)}q^{-a(s)-1})(1-t^{\ell(s)+1}q^{a(s)})}\,, \tag{3.9}$$

where $t=t_1$ and $q^{-1}=t_2$ are equivariant parameters of the toric action. When $q=t$ this generating function agrees with the instanton part of the partition function (3.8),[5] if we identify $y=Q_m$. In the decoupling limit of the adjoint matter $m\to\infty$, the expression is reduced to the generating function of the equivariant Euler character χ_0;

$$\begin{aligned}\sum_{n=0}^\infty Q^n \chi_0\left(\mathrm{Hilb}^n\ \mathbb{C}^2\right)(t,q) &= \sum_\mu Q^{|\mu|}\prod_{s\in\mu}\frac{1}{(1-t^{-\ell(s)}q^{-a(s)-1})(1-t^{\ell(s)+1}q^{a(s)})}\,,\\ &= \exp\left(\sum_{n=1}^\infty \frac{Q^n}{n(1-t^n)(1-q^{-n})}\right).\end{aligned} \tag{3.10}$$

We note that this is identified as the partition function of the abelian gauge theory that is given by the formula (2.8) if we (formally) extend it to U(1) case .

3.2 Two parameter generalization

We will show that the full generating function (3.9) is obtained by topological vertex computation, if we use a two parameter generalization of topological vertex $C_{R_1R_2R_3}(q,t)$ in terms of the Macdonald function $P_R(x;q,t)$. We introduce the refined topological vertex

$$C_{R_1R_2R_3}(q,t) := t^{-n(R_3^t)}q^{n(R_3)}\widetilde{P}_{R_2}(t^{-\rho};q,t)\sum_R \iota\widetilde{P}_{R_1/R}(q^{+\mu^{R_2}}t^{+\rho};q,t)\widetilde{P}_{R_3^t/R}(q^{-\mu^{R_2}}t^{-\rho};q,t)\,, \tag{3.11}$$

and its conjugate

$$C^{R_1R_2R_3}(q,t) := C_{R_1R_2R_3}(t,q)\,, \tag{3.12}$$

[5] The first factor in (3.8) is the perturbative contribution.

where we have exchanged q and t to define the conjugate vertex. We need two types of topological vertices related by the exchange of q and t and will use lower and upper indices to distinguish them. We find it convenient to use a renormalized (skew) Macdonald function $\widetilde{P}_{R/S}(x;q,t)$ defined as follows; Let us introduce the normalization factor

$$f_R^2(q,t) = b_R(q,t) := \prod_{s\in\mu^R} \frac{1-q^{a(s)}t^{\ell(s)+1}}{1-q^{a(s)+1}t^{\ell(s)}}\,. \tag{3.13}$$

We note the relation

$$f_R(q,t)f_{R^t}(t,q) = 1\,. \tag{3.14}$$

Then the renormalized skew Macdonald function is given by

$$\widetilde{P}_{R/S}(x;q,t) := f_R(q,t)f_S^{-1}(q,t)P_{R/S}(x;q,t)\,. \tag{3.15}$$

In (3.11) the notation ι on the second Macdonald function means the involution on the symmetric functions defined by $\iota(p_n) = -p_n$ for the power sum function $p_n(x) = \sum_i x_i^n$. Note that the set of the power sum functions $\{p_n(x)\}_{n\geq 0}$ generates the ring of symmetric functions. Finally $q^\mu t^\rho$ stands for the specialization with $x_i = q^{\mu_i}t^{-i+1/2}$. One may notice that the first index R_1 is distinguished from other indices and we cannot prove the cyclic symmetry of our topological vertex. To construct a refined vertex with nice symmetry is an open problem.

When $q=t$, the (renormalized) Macdonald function is reduced to the Schur function $s_R(x)$. Hence, the refined vertex (3.11) becomes

$$\begin{aligned} C_{R_1R_2R_3}(q) &= C^{R_1R_2R_3}(q) \\ &= q^{n(R_3)-n(R_3^t)}s_{R_2}(q^{-\rho})\sum_R \iota s_{R_1/R}(q^{+\mu^{R_2}+\rho})s_{R_3^t/R}(q^{-\mu^{R_2}-\rho})\,. \end{aligned} \tag{3.16}$$

Using the formula of analytic continuation for the specialization of the Schur function;

$$s_{R/Q}(q^{\mu^t+\rho}) = \iota s_{R/Q}(q^{-\mu-\rho})\,, \tag{3.17}$$

and the relation $\kappa(\mu) = 2(n(\mu^t)-n(\mu))$, we see that (3.16) agrees with the topological vertex in terms of the skew Schur functions [14];

$$C_{R_1R_2R_3}(q) = q^{-\kappa(R_3)/2}s_{R_2}(q^{-\rho})\sum_R s_{R_1/R}(q^{-\mu^{R_2^t}-\rho})s_{R_3^t/R}(q^{-\mu^{R_2}-\rho})\,. \tag{3.18}$$

With the refined topological vertex (3.11) and the same web diagram as before, we have the topological partition function

$$\begin{aligned} Z(Q,Q_m;q,t) &= \sum_{R,R_m}(-Q)^{|\mu^R|}(-Q_m)^{|\mu^{R_m}|}C_{\bullet RR_m}(q,t)C^{\bullet R^tR_m^t}(q,t) \\ &= \sum_R(-Q)^{|\mu^R|}P_R(t^{-\rho};q,t)P_{R^t}(q^{-\rho};t,q)\times \\ &\quad\times\sum_{R_m}(-Q_m)^{|\mu^{R_m}|}P_{R_m^t}(q^{-\mu^R}t^{-\rho};q,t)P_{R_m}(t^{-\mu^{R^t}}q^{-\rho};t,q)\,, \end{aligned} \tag{3.19}$$

where we have used $\widetilde{P}_\mu(x;q,t)\widetilde{P}_{\mu^t}(y;t,q) = P_\mu(x;q,t)P_{\mu^t}(y;t,q)$. Since we can take the first index to be trivial, we can avoid the problem of asymmetry mentioned above in this case. By the specialization formula in appendix B.3 we have

$$\begin{aligned}
P_\lambda(t^{-\rho};q,t) &= \frac{t^{|\lambda|/2+n(\lambda)}}{\prod_{s\in\lambda}(1-q^{a(s)}t^{\ell(s)+1})}\,, \\
P_{\lambda^t}(q^{-\rho};t,q) &= \frac{q^{|\lambda|/2+n(\lambda^t)}}{\prod_{s\in\lambda}(1-t^{\ell(s)}q^{a(s)+1})}\,, \\
&= \frac{(-1)^{|\lambda|}q^{|\lambda|/2+n(\lambda^t)}}{t^{n(\lambda)}q^{n(\lambda^t)+|\lambda|}\prod_{s\in\lambda}(1-t^{-\ell(s)}q^{-a(s)-1})}\,.
\end{aligned} \tag{3.20}$$

Hence we obtain

$$\begin{aligned}
&\sum_R(-Q)^{|\mu^R|}P_R(t^{-\rho};q,t)P_{R^t}(q^{-\rho};t,q) = \\
&\quad = \sum_R Q^{|\mu^R|}\left(\frac{t}{q}\right)^{|\mu^R|/2}\prod_{s\in\mu^R}\frac{1}{(1-q^{a(s)}t^{\ell(s)+1})(1-t^{-\ell(s)}q^{-a(s)-1})}\,.
\end{aligned} \tag{3.21}$$

We next invoke the Cauchy formula to obtain

$$\sum_{R_m}(-Q_m)^{|\mu^R|}P_{R_m^t}(q^{-\mu^R}t^{-\rho};q,t)P_{R_m}(t^{-\mu^{R^t}}q^{-\rho};t,q) = \prod_{i,j\geq 1}\left(1-Q_m q^{-\mu_i^R+j-\frac{1}{2}}t^{-\mu_j^{R^t}+i-\frac{1}{2}}\right). \tag{3.22}$$

Combining two propositions in appendix A.2, we can derive

$$\sum_{s\in\mu}q^{a_\lambda(s)}t^{\ell_\mu(s)+1}+\sum_{s\in\lambda}q^{-a_\mu(s)-1}t^{-\ell_\lambda(s)} = \sum_{i,j\geq 1}\left(q^{-\mu_i}t^{-\lambda_j^t}-1\right)t^i q^{j-1}\,. \tag{3.23}$$

The equation (3.23) with $\mu=\lambda$ implies

$$\begin{aligned}
\prod_{i,j\geq 1}\left(1-Q_m q^{-\mu_i^R+j-\frac{1}{2}}t^{-\mu_j^{R^t}+i-\frac{1}{2}}\right) &= \prod_{i,j\geq 1}\left(1-Q_m q^{j-\frac{1}{2}}t^{i-\frac{1}{2}}\right)\cdot \\
&\quad\cdot\prod_{s\in\mu^R}\left(1-Q_m\left(\frac{q}{t}\right)^{1/2}q^{a(s)}t^{\ell(s)+1}\right)\times \\
&\quad\times\left(1-Q_m\left(\frac{q}{t}\right)^{1/2}t^{-\ell(s)}q^{-a(s)-1}\right).
\end{aligned}$$

Combining all these results we have the generating function (3.9) of the χ_y genus with the identification $y = Q_m\left(\frac{q}{t}\right)^{1/2}$.

3.3 Elliptic genus — six dimensional theory

According to the web-diagram (figure 2) of the six dimensional theory, the topological partition function is

$$\begin{aligned}
Z(Q,Q_v,Q_m;q,t) &= \sum_{R,R_v,R_m}(-Q)^{|\mu^R|}(-Q_v)^{|\mu^{R_v}|}(-Q_m)^{|\mu^{R_m}|}C_{R_vRR_m}(q,t)C^{R_v^tR^tR_m^t}(q,t) \\
&= \sum_R(-Q)^{|\mu^R|}P_R(t^{-\rho};q,t)P_{R^t}(q^{-\rho};t,q)Z_R(Q_v,Q_m;q,t)\,,
\end{aligned} \tag{3.24}$$

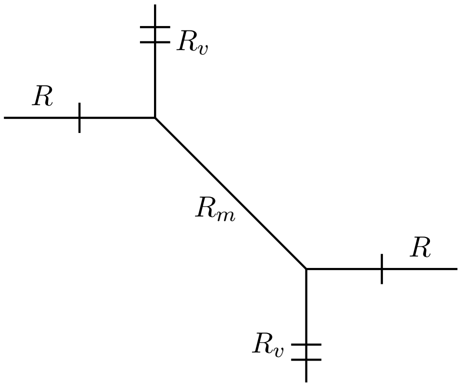

Figure 2: Geometric engineering of six dimensional U(1) theory with adjoint hypermultiplet. In addition to the horizontal external lines the vertical external lines (NS 5-branes) are identified.

where

$$\begin{aligned} Z_R(Q_v, Q_m; q, t) = & \sum_{R_v, R_m} (-Q_v)^{|\mu^{R_v}|} (-Q_m)^{|\mu^{R_m}|} \times \\ & \times \sum_{T_1, T_2} \iota \widetilde{P}_{R_v/T_1}(q^{\mu^R} t^{\rho}; q, t) \widetilde{P}_{R_m^t/T_1}(q^{-\mu^R} t^{-\rho}; q, t) \times \\ & \times \iota \widetilde{P}_{R_v^t/T_2}(t^{\mu^{R^t}} q^{\rho}; t, q) \widetilde{P}_{R_m/T_2}(t^{-\mu^{R^t}} q^{-\rho}; t, q) \,. \end{aligned} \tag{3.25}$$

We note that the prefactor $t^{-n(R_3^t)} q^{n(R_3)}$ in our definition (3.11) of the refined topological vertex plays no role here. We expect this factor is important when we consider the web diagram with non-trivial framing.

In the following computation of Z_R, we will repeatedly employ the Cauchy formula for the skew Macdonald function;

$$\sum_R \widetilde{P}_{R/R_1}(x; q, t) \widetilde{P}_{R/R_2}(y; q, t) = \Pi(x, y; q, t) \sum_S \widetilde{P}_{R_2/S}(x; q, t) \widetilde{P}_{R_1/S}(y; q, t) \,, \tag{3.26}$$

$$\sum_R \widetilde{P}_{R/R_1}(x; q, t) \widetilde{P}_{R^t/R_2^t}(y; t, q) = \Pi_0(x, y) \sum_S \widetilde{P}_{R_2/S}(x; q, t) \widetilde{P}_{R_1^t/S^t}(y; t, q) \,, \tag{3.27}$$

where

$$\Pi(x, y; q, t) := \prod_{i,j \geq 1} \frac{(t x_i y_j; q)_\infty}{(x_i y_j; q)_\infty} = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{1 - t^n}{1 - q^n} p_n(x) p_n(y) \right), \tag{3.28}$$

$$\Pi_0(x, y) := \prod_{i,j \geq 1} (1 + x_i y_j) = \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} p_n(x) p_n(y) \right). \tag{3.29}$$

Recall the definition of the involution ι; $\iota(p_n) = -p_n$. Looking at (3.28) and (3.29), we see the following Cauchy formula with involution;

$$\sum_R \iota \widetilde{P}_{R/R_1}(x; q, t) \widetilde{P}_{R/R_2}(y; q, t) = \Pi(x, y; q, t)^{-1} \sum_S \iota \widetilde{P}_{R_2/S}(x; q, t) \widetilde{P}_{R_1/S}(y; q, t) \,, \tag{3.30}$$

$$\sum_R \iota\widetilde{P}_{R/R_1}(x;q,t)\iota\widetilde{P}_{R^t/R_2^t}(y;t,q) = \Pi_0(x,y)\sum_S \iota\widetilde{P}_{R_2/S}(x;q,t)\iota\widetilde{P}_{R_1^t/S^t}(y;t,q)\,. \tag{3.31}$$

Now we are ready for computing Z_R. By the Cauchy formula, we have

$$\begin{aligned}
Z_R(Q_v,Q_m:q,t) &= \prod_{i,j\geq 1}\left(1-Q_v q^{\mu_i^R-j+\frac{1}{2}}t^{\mu_j^{R^t}-i+\frac{1}{2}}\right)\left(1-Q_m q^{-\mu_i^R+j-\frac{1}{2}}t^{-\mu_j^{R^t}+i-\frac{1}{2}}\right)\times\\
&\quad \sum_{S_1S_2}(-Q_v)^{|\mu^{S_1}|}(-Q_m)^{|\mu^{S_2}|}\sum_{T_1T_2}\iota\widetilde{P}_{T_2^t/S_1}(Q_v q^{\mu^R}t^{\rho};q,t)\times\\
&\quad \times\iota\widetilde{P}_{T_1^t/S_1^t}(Q_v t^{\mu^{R^t}}q^{\rho};t,q)\widetilde{P}_{T_2^t/S_2}(Q_m q^{-\mu^R}t^{-\rho};q,t)\times\\
&\quad \times\widetilde{P}_{T_1^t/S_2^t}(Q_m t^{-\mu^{R^t}}q^{-\rho};t,q)\\
&= \prod_{i,j\geq 1}\left(1-Q_v q^{\mu_i^R-j+\frac{1}{2}}t^{\mu_j^{R^t}-i+\frac{1}{2}}\right)\left(1-Q_m q^{-\mu_i^R+j-\frac{1}{2}}t^{-\mu_j^{R^t}+i-\frac{1}{2}}\right)\times\\
&\quad \times\Pi^{-1}\left(Q_v q^{\mu^R}t^{\rho},Q_m q^{-\mu^R}t^{-\rho};q,t\right)\times\\
&\quad \times\Pi^{-1}\left(Q_v t^{\mu^{R^t}}q^{\rho},Q_m t^{-\mu^{R^t}}q^{-\rho};t,q\right)\times\\
&\quad \times\sum_{S_1S_2}(-Q_v)^{|\mu^{S_1}|}(-Q_m)^{|\mu^{S_2}|}\sum_{T_3T_4}\iota\widetilde{P}_{S_2/T_3}(Q_v q^{\mu^R}t^{\rho};q,t)\times\\
&\quad \times\widetilde{P}_{S_1/T_3}(Q_m q^{-\mu^R}t^{-\rho};q,t)\iota\widetilde{P}_{S_2^t/T_4}(Q_v t^{\mu^{R^t}}q^{\rho};t,q)\times\\
&\quad \times\widetilde{P}_{S_1^t/T_4}(Q_m t^{-\mu^{R^t}}q^{-\rho};t,q)\,.
\end{aligned} \tag{3.32}$$

Our rule of analytic continuation explained in appendix A.2 implies

$$\begin{aligned}
\Pi(q^{\lambda}t^{\rho},q^{-\mu}t^{-\rho};q,t) &= \Pi_0(-t^{-\lambda^t}q^{-\rho-\frac{1}{2}},q^{-\mu}t^{-\rho+\frac{1}{2}})\\
&= \Pi_0(q^{\lambda}t^{\rho+\frac{1}{2}},-t^{\mu^t}q^{\rho-\frac{1}{2}})\,.
\end{aligned} \tag{3.33}$$

Hence we obtain

$$\begin{aligned}
Z_R &= \prod_{i,j\geq 1}\frac{\left(1-Q_v q^{\mu_i^R-j+\frac{1}{2}}t^{\mu_j^{R^t}-i+\frac{1}{2}}\right)\left(1-Q_m q^{-\mu_i^R+j-\frac{1}{2}}t^{-\mu_j^{R^t}+i-\frac{1}{2}}\right)}{\left(1-Q_vQ_m q^{\mu_i^R-j}t^{\mu_j^{R^t}-i+1}\right)\left(1-Q_vQ_m q^{-\mu_i^R+j}t^{-\mu_j^{R^t}+i-1}\right)}\times\\
&\quad \times\sum_{S_1S_2}(-Q_v)^{|\mu^{S_1}|}(-Q_m)^{|\mu^{S_2}|}\sum_{T_3T_4}\iota\widetilde{P}_{S_2/T_3}(Q_v q^{\mu^R}t^{\rho};q,t)\times\\
&\quad \times\widetilde{P}_{S_1/T_3}(Q_m q^{-\mu^R}t^{-\rho};q,t)\iota\widetilde{P}_{S_2^t/T_4}(Q_v t^{\mu^{R^t}}q^{\rho};t,q)\widetilde{P}_{S_1^t/T_4}(Q_m t^{-\mu^{R^t}}q^{-\rho};t,q)\,.
\end{aligned} \tag{3.34}$$

Here we notice that the sum over the representations is the same as before except that the arguments of $\iota\widetilde{P}$ and $\widetilde{P}$ are multiplied by Q_v and Q_m, respectively. Thus we can repeat the same procedure of using the Cauchy formula. By iteration we finally obtain

$$Z_R = \prod_{k=1}^{\infty}\prod_{i,j\geq 1}\frac{\left(1-Q_v^kQ_m^{k-1}q^{\mu_i^R-j+\frac{1}{2}}t^{\mu_j^{R^t}-i+\frac{1}{2}}\right)\left(1-Q_v^{k-1}Q_m^k q^{-\mu_i^R+j-\frac{1}{2}}t^{-\mu_j^{R^t}+i-\frac{1}{2}}\right)}{\left(1-Q_v^kQ_m^k q^{\mu_i^R-j}t^{\mu_j^{R^t}-i+1}\right)\left(1-Q_v^kQ_m^k q^{-\mu_i^R+j}t^{-\mu_j^{R^t}+i-1}\right)}\,. \tag{3.35}$$

The formula (3.23) with $\lambda = \mu$

$$\sum_{s\in\mu}\left(q^{a(s)}t^{\ell(s)+1} + q^{-a(s)-1}t^{-\ell(s)}\right) = \sum_{i,j\geq 1}\left(q^{-\mu_i}t^{-\mu_j^t} - 1\right)t^i q^{j-1}\,, \tag{3.36}$$

$$\sum_{s\in\mu}\left(q^{-a(s)}t^{-\ell(s)-1} + q^{a(s)+1}t^{\ell(s)}\right) = \sum_{i,j\geq 1}\left(q^{\mu_i}t^{\mu_j^t} - 1\right)t^{-i} q^{-j+1}\,, \tag{3.37}$$

allows us to convert the infinite product over two indices i, j into the finite product over the boxes of the Young diagram;

$$\begin{aligned} Z_R = Z_{\text{perturb}} \cdot \prod_{k=1}^{\infty}\prod_{s\in\mu^R} & \frac{\left(1 - Q_v^k Q_m^{k-1}\left(\frac{t}{q}\right)^{\frac{1}{2}} q^{-a(s)}t^{-\ell(s)-1}\right)\left(1 - Q_v^k Q_m^{k-1}\left(\frac{t}{q}\right)^{\frac{1}{2}} q^{a(s)+1}t^{\ell(s)}\right)}{(1 - Q_v^k Q_m^k q^{-a(s)-1}t^{-\ell(s)})(1 - Q_v^k Q_m^k q^{a(s)}t^{\ell(s)+1})} \times \\ & \times \frac{\left(1 - Q_v^{k-1} Q_m^{k}\left(\frac{q}{t}\right)^{\frac{1}{2}} q^{a(s)}t^{\ell(s)+1}\right)\left(1 - Q_v^{k-1} Q_m^{k}\left(\frac{q}{t}\right)^{\frac{1}{2}} q^{-a(s)-1}t^{-\ell(s)}\right)}{(1 - Q_v^k Q_m^k q^{a(s)+1}t^{\ell(s)})(1 - Q_v^k Q_m^k q^{-a(s)}t^{-\ell(s)-1})}\,, \end{aligned} \tag{3.38}$$

where

$$Z_{\text{perturb}} = \prod_{k=1}^{\infty}\prod_{i,j\geq 1}\frac{\left(1 - Q_v^k Q_m^{k-1} q^{-j+\frac{1}{2}}t^{-i+\frac{1}{2}}\right)\left(1 - Q_v^{k-1} Q_m^{k} q^{j-\frac{1}{2}}t^{i-\frac{1}{2}}\right)}{(1 - Q_v^k Q_m^k q^{-j}t^{-i+1})(1 - Q_v^k Q_m^k q^{j}t^{i-1})}\,. \tag{3.39}$$

and we have neglected the factor $\prod_{k=1}^{\infty}(1 - Q_v^k Q_m^k)^{-1}$ that is independent of q and t. Combined with the contribution of (3.21), the instanton part of Z is identified as follows;

$$\begin{aligned} Z(Q,y,p;q,t) = \sum_R Q^{|\mu^R|}\left(\frac{t}{q}\right)^{|\mu^R|/2}\prod_{k=1}^{\infty}\prod_{s\in\mu^R} & \frac{\left(1 - p^k y^{-1} q^{-a(s)}t^{-\ell(s)-1}\right)\left(1 - p^k y^{-1} q^{a(s)+1}t^{\ell(s)}\right)}{\left(1 - p^{k-1} q^{-a(s)-1}t^{-\ell(s)}\right)\left(1 - p^{k-1} q^{a(s)}t^{\ell(s)+1}\right)} \times \\ & \times \frac{\left(1 - p^{k-1} y q^{a(s)}t^{\ell(s)+1}\right)\left(1 - p^{k-1} y q^{-a(s)-1}t^{-\ell(s)}\right)}{\left(1 - p^{k} q^{a(s)+1}t^{\ell(s)}\right)\left(1 - p^{k} q^{-a(s)}t^{-\ell(s)-1}\right)}\,, \end{aligned} \tag{3.40}$$

where $p := Q_v Q_m$ and $y := Q_m \cdot \left(\frac{q}{t}\right)^{\frac{1}{2}}$. We can see the final result agrees to the generating function of the elliptic genus of $\text{Hilb}^n\, \mathbb{C}^2$ given in [44] with $t = t_1^{-1}$ and $q = t_2$.

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A. Formula for partition

A.1 Arm-length, leg length and related quantities

For each square $s = (i, j)$ in the Young diagram of a partition $\{\lambda_i\}$, we define

$$a_\lambda(s) := \lambda_i - j\,, \qquad \ell_\lambda(s) := \lambda_j^\vee - i\,, \qquad a'(s) := j - 1\,, \qquad \ell'(s) := i - 1\,, \tag{A.1}$$

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where $\lambda_j^\vee$ denotes the conjugate (dual) diagram. They are called arm-length, leg-length, arm-colength and leg-colength, respectively. The hook length $h_\lambda(s)$ and the content $c(s)$ at s are given by

$$h_\lambda(s) = a_\lambda(s) + \ell_\lambda(s) + 1\,, \qquad c(s) = a'(s) - \ell'(s)\,. \tag{A.2}$$

We also need the following integer

$$n(\lambda) := \sum_{s\in\lambda} \ell'(s) = \sum_{i=1}^{\infty} (i-1)\lambda_i = \frac{1}{2}\sum_{i=1}^{\infty} \lambda_i^\vee(\lambda_i^\vee - 1) = \sum_{s\in\lambda} \ell_\lambda(s)\,. \tag{A.3}$$

Similarly we have

$$n(\lambda^\vee) := \sum_{s\in\lambda} a'(s) = \sum_{s\in\lambda} a_\lambda(s)\,. \tag{A.4}$$

They are related to the integer $\kappa(\lambda)$ as follows;

$$\kappa(\lambda) := 2\sum_{s\in\lambda}(j-i) = 2(n(\lambda^\vee) - n(\lambda)) = |\lambda| + \sum_{i=1}^{\infty} \lambda_i(\lambda_i - 2i)\,. \tag{A.5}$$

A.2 Combinatorial identities

Next we will prove two useful propositions. First, we have

Lemma 1. *For all integers $N \geq \ell(\lambda)$*

$$(1-q)\sum_{(i,j)\in\lambda} q^{j-1}t^{-i+1} = \sum_{i=1}^{N}\left(1 - q^{\lambda_i}\right)t^{-i+1}\,, \tag{A.6}$$

$$\begin{aligned}(1-q)\sum_{(i,j)\in\mu} q^{\lambda_i - j}t^{\mu_j^\vee - i + 1} = (t-1)\sum_{1\leq i<j\leq N+1}\left(q^{\lambda_i-\mu_j} - 1\right)t^{j-i} + t\sum_{i=1}^{N}\left(q^{\lambda_i-\mu_i} - 1\right) - \\ - \sum_{i=1}^{N}\left(q^{\lambda_i} - 1\right)t^{N-i+2}\,.\end{aligned} \tag{A.7}$$

The former in this lemma is nothing but $\sum_{j=1}^{\lambda} q^{j-1} = (1-q^\lambda)/(1-q)$. The proof of the latter is similar to that in [30].

When $|t| < 1$, the last term of (A.7) vanishes as N tends to infinity. Using this formula and that replacing q, t and λ with $1/q$, $1/t$ and μ, respectively, we obtain

Proposition 1.

$$\sum_{s\in\mu} q^{a_\lambda(s)}t^{\ell_\mu(s)+1} + \sum_{s\in\lambda} q^{-a_\mu(s)-1}t^{-\ell_\lambda(s)} = \frac{t-1}{1-q}\sum_{i,j=1}^{\infty}\left(q^{\lambda_i-\mu_j} - 1\right)t^{j-i}\,. \tag{A.8}$$

By the exchange $(\lambda,\mu)\to(\lambda^\vee,\mu^\vee)$ and $(q,t)\to(t,q)$, we obtain the transposed version;

$$\sum_{s\in\mu} q^{a_\mu(s)+1}t^{\ell_\lambda(s)}+\sum_{s\in\lambda} q^{-a_\lambda(s)}t^{-\ell_\mu(s)-1}=\frac{q-1}{1-t}\sum_{i,j=1}^{\infty}\left(t^{\lambda_i^\vee-\mu_j^\vee}-1\right)q^{j-i}. \tag{A.9}$$

In this formula we are aware of the problem of the domain of the convergence of the geometric series in t. We understand that the geometric series is computed in an appropriate domain in the complex t-plane and then analytically continued to the whole plane as rational function with a pole at $t=1$ (and $q=1$).

Next, similar to (A.6), we have

Lemma 2. *For all integers $N\geq\ell(\lambda)$ and $M\geq\lambda_1$,*

$$\left(t^{\frac{1}{2}}-t^{-\frac{1}{2}}\right)\sum_{i=1}^{N} q^{\lambda_i}t^{\frac{1}{2}-i}+\left(q^{\frac{1}{2}}-q^{-\frac{1}{2}}\right)\sum_{i=1}^{M} t^{-\lambda_i^\vee}q^{i-\frac{1}{2}}=q^M-t^{-N}. \tag{A.10}$$

Where $\ell(\lambda)$ is the length of the partition λ, which is the non-zero numbers of parts λ_i's.

When $|q|$, $|t^{-1}|<1$, the right hand side of (A.11) vanishes as N and M tends to infinity. Hence we obtain

Proposition 2.

$$(t-1)\sum_{i=1}^{\infty} q^{\lambda_i}t^{-i}=(q^{-1}-1)\sum_{i=1}^{\infty} t^{-\lambda_i^\vee}q^{i}. \tag{A.11}$$

Taking $(q,t)\to(q^{-1},t^{-1})$ we have the analytically continued version;

$$(t^{-1}-1)\sum_{i=1}^{\infty} q^{-\lambda_i}t^{i}=(q-1)\sum_{i=1}^{\infty} t^{\lambda_i^\vee}q^{-i}. \tag{A.12}$$

When $q=t$ these propositions become

$$\sum_{s\in\mu} q^{a_\lambda(s)+\ell_\mu(s)+1}+\sum_{s\in\lambda} q^{-a_\mu(s)-\ell_\lambda(s)-1}=-\sum_{1\leq i,j<\infty}\left(q^{\lambda_i-\mu_j+j-i}-q^{j-i}\right), \tag{A.13}$$

$$\sum_{s\in\mu} q^{a_\mu(s)+\ell_\lambda(s)+1}+\sum_{s\in\lambda} q^{-a_\lambda(s)-\ell_\mu(s)-1}=-\sum_{1\leq i,j<\infty}\left(q^{\lambda_i^\vee-\mu_j^\vee+j-i}-q^{j-i}\right), \tag{A.14}$$

$$(q-1)\sum_{i} q^{\lambda_i-i}=(q^{-1}-1)\sum_{i} q^{i-\lambda_i^\vee}. \tag{A.15}$$

The last equality can be rewritten as

$$-\sum_{i=1}^{\infty} q^{\lambda_i-i+\frac{1}{2}}=\sum_{i=1}^{\infty} q^{-\lambda_i^\vee+i-\frac{1}{2}}, \qquad -\sum_{i=1}^{\infty} q^{-i+\frac{1}{2}}=\sum_{i=1}^{\infty} q^{i-\frac{1}{2}}. \tag{A.16}$$

This formula may be regarded as our rule of analytic continuation.

By using these propositions 1 and 2, one can derive the following relations, respectively

$$
\prod_{(i,j)\in\mu} \frac{1}{1-Q\,q^{\lambda_i-j}t^{\mu_j^\vee-i+1}} \cdot \prod_{(i,j)\in\lambda} \frac{1}{1-Q\,q^{-\mu_i+j-1}t^{-\lambda_j^\vee+i}} = \frac{\Pi\left(Q\,t^\rho, t^{-\rho}\right)}{\Pi\left(Q\,q^\lambda t^\rho, q^{-\mu}t^{-\rho}\right)}\,, \qquad Q\in\mathbb{C}
$$

$$
\Pi\left(Q\,q^\lambda t^\rho,\; q^{-\mu}t^{-\rho}\right) = \begin{cases} \Pi_0\left(-Q\left(\dfrac{t}{q}\right)^{\frac{1}{2}} t^{-\lambda^\vee} q^{-\rho},\; q^{-\mu}t^{-\rho}\right), \\ \Pi_0\left(-Q\left(\dfrac{t}{q}\right)^{\frac{1}{2}} q^{\lambda}t^{\rho},\; t^{\mu^\vee}q^{\rho}\right), \end{cases} \tag{A.17}
$$

Where $\Pi(x,y)$ and $\Pi_0(x,y)$ are the Cauchy kernel and its conjugate in (B.20) and (B.21), respectively. These represent the equivalence between several expressions of the Nekrasov formula in [26] and [27].

B. Formula for the Macdonald symmetric function

In this appendix we recapitulate basic properties of the Macdonald symmetric function [30].

B.1 Definition for the Macdonald symmetric function

There are various bases of the ring of symmetric functions in infinite number of variables $x=(x_1,x_2,\ldots)$, for example, the monomial symmetric function, the power-sum symmetric function and so on. They are indexed by the Young diagram, i.e., the partition $\lambda = (\lambda_1,\lambda_2,\ldots)$, which is a sequence of non-negative integers such that $\lambda_i \geq \lambda_{i+1}$ and $|\lambda| = \sum_i \lambda_i < \infty$. The monomial symmetric function $m_\lambda(x)$ is defined by

$$
m_\lambda(x) = \sum_\sigma x_1^{\lambda_{\sigma(1)}} x_2^{\lambda_{\sigma(2)}} \ldots\,, \tag{B.1}
$$

where the summation is over all distinct permutations of $(\lambda_1,\lambda_2,\ldots)$.

The power-sum symmetric function $p_\lambda(x)$ is defined by

$$
p_\lambda(x) = p_{\lambda_1}(x)p_{\lambda_2}(x)\ldots\,, \qquad p_n(x) = \sum_{i=1}^\infty x_i^n\,. \tag{B.2}
$$

We introduce an inner-product on the ring of symmetric functions in the following manner; for any symmetric functions f and g,

$$
\langle f(p),\, g(p)\rangle_{q,t} := f(p^*)\, g(p)\,|_{\text{constant part}}\,, \qquad p_n^* := n\frac{1-q^n}{1-t^n}\frac{\partial}{\partial p_n}\,, \tag{B.3}
$$

or equivalently

$$
\langle p_\lambda, p_\mu\rangle_{q,t} = \delta_{\lambda,\mu}\prod_{r\geq 1} r^{m_r} m_r! \cdot \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}\,, \qquad \lambda = (1^{m_1}2^{m_2}\ldots)\,, \tag{B.4}
$$

with $m_r \equiv \#\{i\,|\,\lambda_i = r\}$.

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The Macdonald symmetric function $P_\lambda = P_\lambda(x;q,t)$ is uniquely specified by the following orthogonality and normalization,

$$\langle P_\lambda, P_\mu \rangle_{q,t} = 0 \qquad \text{if } \lambda \neq \mu \,, \tag{B.5}$$

$$P_\lambda = m_\lambda + \sum_{\mu<\lambda} u_{\lambda\mu} m_\mu \,, \quad u_{\lambda\mu} \in \mathbf{Q}(q,t) \,. \tag{B.6}$$

Here we used the dominance partial ordering on the Young diagrams defined as $\lambda \geq \mu \Leftrightarrow |\lambda| = |\mu|$ and $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$ for all i.

The first few examples are

$$\begin{aligned} P_1 &= m_1 , \\ \begin{bmatrix} P_2 \\ P_{1,1} \end{bmatrix} &= \begin{bmatrix} 1 & \frac{(1+q)(1-t)}{1-qt} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m_2 \\ m_{1,1} \end{bmatrix} , \\ \begin{bmatrix} P_3 \\ P_{2,1} \\ P_{1,1,1} \end{bmatrix} &= \begin{bmatrix} 1 & \frac{(1+q+q^2)(1-t)}{1-q^2 t} & \frac{(1+q)(1+q+q^2)(1-t)^2}{(1-qt)(1-q^2t)} \\ 0 & 1 & \frac{(2+q+t+2qt)(1-t)}{1-qt^2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_3 \\ m_{2,1} \\ m_{1,1,1} \end{bmatrix} . \end{aligned} \tag{B.7}$$

The Macdonald symmetric function contains several important symmetric functions as a case of special values of q and t. For example,

(i) $P_\lambda(x;q,q) = s_\lambda(x)$; the Schur function.

(ii) $P_\lambda(x;0,t) = P_\lambda(x;t)$; the Hall-Littlewood function.

(iii) $\lim_{q\to 1} P_\lambda(x;q,q^\beta) = P_\lambda^{(1/\beta)}(x)$; the Jack symmetric function.

(iv) $P_\lambda(x;q,1) = m_\lambda(x)$; the monomial symmetric function.

(v) $P_\lambda(x;1,t) = e_{\lambda^\vee}(x)$; the elementary symmetric function.

The scalar-product is given by

$$\langle P_\lambda | P_\lambda \rangle_{q,t} = \prod_{s\in\lambda} \frac{1-q^{a(s)+1}t^{\ell(s)}}{1-q^{a(s)}t^{\ell(s)+1}} \,, \tag{B.8}$$

which satisfies

$$\langle P_\lambda | P_\lambda \rangle_{q,t} \langle P_{\lambda^\vee} | P_{\lambda^\vee} \rangle_{t,q} = 1 \,, \tag{B.9}$$

$$\frac{(q/t)^{\frac{|\lambda|}{2}}}{\langle P_\lambda | P_\lambda \rangle_{q,t}} = \frac{(t/q)^{\frac{|\lambda|}{2}}}{\langle P_\lambda | P_\lambda \rangle_{q^{-1},t^{-1}}} \,. \tag{B.10}$$

Let $\widetilde{P}_\lambda(x;q,t)$ be the normalized Macdonald function

$$\widetilde{P}_\lambda(x;q,t) := \frac{1}{\langle P_\lambda | P_\lambda \rangle_{q,t}^{\frac{1}{2}}} P_\lambda(x;q,t) \,, \tag{B.11}$$

so that the scalar product is normalized as $\langle \widetilde{P}_\lambda | \widetilde{P}_\lambda \rangle_{q,t} = 1$.

The normalized skew-Macdonald symmetric function $\widetilde{P}_{\lambda/\mu}(x;q,t)$ is defined by

$$\widetilde{P}_{\lambda/\mu}(x;q,t) := \widetilde{P}^{*}_{\mu}(x;q,t)\ \widetilde{P}_{\lambda}(x;q,t)\,, \tag{B.12}$$

where $*$ is acted on the power-sum as $p_n^* := n\frac{1-q^n}{1-t^n}\frac{\partial}{\partial p_n}$. The relation with the usual skew-Macdonald function $P_{\lambda/\mu}(x;q,t)$ is

$$\widetilde{P}_{\lambda/\mu}(x;q,t) := \frac{\langle P_\mu|P_\mu\rangle_{q,t}^{\frac{1}{2}}}{\langle P_\lambda|P_\lambda\rangle_{q,t}^{\frac{1}{2}}} P_{\lambda/\mu}(x;q,t)\,. \tag{B.13}$$

Finally let $\iota\widetilde{P}_{\lambda/\mu}(x;q,t)$ be the skew-Macdonald function with the involution ι acting on the power-sum p_n as $\iota(p_n) = -p_n$.

B.2 Symmetries and Cauchy formulas

The skew-Macdonald symmetric function enjoys the following symmetries

$$\begin{aligned}
\widetilde{P}_{\lambda/\mu}(Qx;q,t) &= Q^{|\lambda|-|\mu|}\widetilde{P}_{\lambda/\mu}(x;q,t)\,, && \text{(B.14)}\\
\widetilde{P}_{\lambda/\mu}\left(x;q^{-1},t^{-1}\right) &= \left(\frac{q}{t}\right)^{\frac{|\lambda|-|\mu|}{2}}\widetilde{P}_{\lambda/\mu}(x;q,t)\,, && \text{(B.15)}\\
\widetilde{P}_{\lambda^\vee/\mu^\vee}(x;t,q) &= \omega_{q,t}\widetilde{P}_{\lambda/\mu}(x;q,t)\,, && \text{(B.16)}
\end{aligned}$$

with the endmorphism $\omega_{q,t}$ such that

$$\omega_{q,t}(p_n) = (-1)^{n-1}\frac{1-q^n}{1-t^n}p_n\,. \tag{B.17}$$

Note that

$$\widetilde{P}_{\lambda/\mu}(x;q,t)\widetilde{P}_{\lambda^\vee/\mu^\vee}(y;t,q) = P_{\lambda/\mu}(x;q,t)P_{\lambda^\vee/\mu^\vee}(y;t,q)\,. \tag{B.18}$$

In the $t=q$ case, the Schur function satisfies also

$$s_{\lambda^\vee}(x) = \iota s_\lambda(-x) = (-1)^{|\lambda|}\iota s_\lambda(x)\,. \tag{B.19}$$

The following Cauchy formula is especially important;

$$\begin{aligned}
\sum_\lambda \widetilde{P}_\lambda(x;q,t)\widetilde{P}_\lambda(y;q,t) &= \Pi(x,y)\,,\\
&:= \prod_{k\geq 0}\prod_{i,j}\frac{1-tx_iy_jq^k}{1-x_iy_jq^k}\,, \qquad |q|<1\,,\\
&= \exp\left\{\sum_{n>0}\frac{1}{n}\frac{1-t^n}{1-q^n}p_n(x)p_n(y)\right\}. && \text{(B.20)}
\end{aligned}$$

By acting on the variables y with the endmorphism $\omega_{q,t}$ we get

$$\sum_\lambda \widetilde{P}_\lambda(x;q,t)\widetilde{P}_{\lambda^\vee}(y;t,q) = \Pi_0(x,y)\,,$$

$$
\begin{aligned}
&:= \prod_{i,j}(1+x_i y_j)\,,\\
&= \exp\left\{\sum_{n>0}\frac{(-1)^{n-1}}{n}p_n(x)p_n(y)\right\}.
\end{aligned}
\tag{B.21}
$$

If we act with the involution ι, then $\Pi(x,y)$ and $\Pi_0(x,y)$ are mapped to their inverse,

$$
\sum_\lambda \widetilde{P}_\lambda(x;q,t)\ \iota\widetilde{P}_\lambda(y;q,t) = \Pi(x,y)^{-1}\,, \tag{B.22}
$$

$$
\sum_\lambda \widetilde{P}_\lambda(x;q,t)\ \iota\widetilde{P}_{\lambda^\vee}(y;t,q) = \Pi_0(x,y)^{-1}\,. \tag{B.23}
$$

The Cauchy formulas for the skew-Macdonald function are

$$
\sum_\lambda \widetilde{P}_{\lambda/\mu}(x;q,t)\widetilde{P}_{\lambda/\nu}(y;q,t) = \Pi(x,y)\sum_\lambda \widetilde{P}_{\mu/\lambda}(y;q,t)\widetilde{P}_{\nu/\lambda}(x;q,t)\,, \tag{B.24}
$$

$$
\sum_\lambda \widetilde{P}_{\lambda/\mu}(x;q,t)\widetilde{P}_{\lambda^\vee/\nu^\vee}(y;t,q) = \Pi_0(x,y)\sum_\lambda \widetilde{P}_{\mu^\vee/\lambda^\vee}(y;t,q)\widetilde{P}_{\nu/\lambda}(x;q,t)\,. \tag{B.25}
$$

Using this formula successively, we obtain the following trace formula

Proposition 3. For $|a|,|b|<1$,

$$
\begin{aligned}
&\sum_{\lambda,\mu,\nu,\rho} a^{|\lambda|}b^{|\nu|}\widetilde{P}_{\lambda/\mu}(x;q,t)\widetilde{P}_{\nu/\mu}(y;q,t)\widetilde{P}_{\nu/\rho}(z;q,t)\widetilde{P}_{\lambda/\rho}(w;q,t) =\\
&\quad = \prod_{k>0}\Pi(a^kb^kx,y)\,\Pi(a^kb^kz,w)\,\Pi(a^kb^{k-1}x,w)\,\Pi(a^{k-1}b^kz,y)\,(1-a^kb^k)^{-1}\,.
\end{aligned}
\tag{B.26}
$$

B.3 Specialization formulas

Here we give the specialization formulas for the special variables. In the case of $|t|>1$, one can set the variables x to the principal specialization $x=t^\rho$ that stands for $x_i = t^{\frac{1}{2}-i}$. The following are the formulas for the principal specialization [30]

$$
\begin{aligned}
P_\lambda(t^\rho;q,t) &= \prod_{s\in\lambda}\frac{-t^{\frac12}q^{a(s)}}{1-q^{a(s)}t^{\ell(s)+1}}\,, \quad \iota P_\lambda(t^\rho;q,t) = \prod_{s\in\lambda}\frac{t^{\frac12}t^{\ell(s)}}{1-q^{a(s)}t^{\ell(s)+1}}\,,\ \ |t^{-1}|<1\,,\\
P_\lambda(t^{-\rho};q,t) &= \prod_{s\in\lambda}\frac{t^{\frac12}t^{\ell(s)}}{1-q^{a(s)}t^{\ell(s)+1}}\,, \quad \iota P_\lambda(t^{-\rho};q,t) = \prod_{s\in\lambda}\frac{-t^{\frac12}q^{a(s)}}{1-q^{a(s)}t^{\ell(s)+1}}\,,\ \ |t|<1\,.
\end{aligned}
\tag{B.27}
$$

$$
\begin{aligned}
P_{\lambda^\vee}(q^\rho;t,q) &= \prod_{s\in\lambda}\frac{q^{-\frac12}q^{-a(s)}}{1-q^{-a(s)-1}t^{-\ell(s)}}\,, \quad \iota P_{\lambda^\vee}(q^\rho;t,q) = \prod_{s\in\lambda}\frac{-q^{-\frac12}t^{-\ell(s)}}{1-q^{-a(s)-1}t^{-\ell(s)}}\,,\ \ |q^{-1}|<1\,,\\
P_{\lambda^\vee}(q^{-\rho};t,q) &= \prod_{s\in\lambda}\frac{-q^{-\frac12}t^{-\ell(s)}}{1-q^{-a(s)-1}t^{-\ell(s)}}\,, \quad \iota P_{\lambda^\vee}(q^{-\rho};t,q) = \prod_{s\in\lambda}\frac{q^{-\frac12}q^{-a(s)}}{1-q^{-a(s)-1}t^{-\ell(s)}}\,,\ \ |q|<1\,.
\end{aligned}
\tag{B.28}
$$

By using the formula of the analytic continuation (A.11), we have

Proposition 4.

$$\widetilde{P}_{\lambda^\vee/\mu^\vee}\left(t^{\pm\eta^\vee}q^{\pm\rho};t,q\right) = (-1)^{|\lambda|-|\mu|}\left(\frac{q}{t}\right)^{\frac{|\lambda|-|\mu|}{2}}\widetilde{P}_{\lambda/\mu}\left(q^{\mp\eta}t^{\mp\rho};q,t\right), \qquad |q^{\mp 1}|,|t^{\pm 1}|<1\,. \tag{B.29}$$

Here $q^\eta t^\rho$ stands for $x_i = q^{\eta_i}t^{\frac{1}{2}-i}$. In the principal case

$$\widetilde{P}_{\lambda^\vee}\left(q^\rho;t,q\right) = \widetilde{P}_\lambda\left(t^{-\rho};q,t\right)(-1)^{|\lambda|}\left(\frac{q}{t}\right)^{\frac{|\lambda|}{2}}, \tag{B.30}$$

$$= \iota\widetilde{P}_\lambda\left(t^\rho;q,t\right)(-1)^{|\lambda|}\left(\frac{q}{t}\right)^{\frac{|\lambda|}{2}}, \tag{B.31}$$

$$= \widetilde{P}_\lambda\left(t^\rho;q,t\right)\left(\frac{q}{t}\right)^{\frac{|\lambda|}{2}}\prod_{s\in\lambda}q^{-a(s)}t^{\ell(s)}\,. \tag{B.32}$$

References

[1] R. Gopakumar and C. Vafa, *Topological gravity as large-N topological gauge theory*, *Adv. Theor. Math. Phys.* **2** (1998) 413 [hep-th/9802016].

[2] R. Gopakumar and C. Vafa, *On the gauge theory/geometry correspondence*, *Adv. Theor. Math. Phys.* **3** (1999) 1415 [hep-th/9811131].

[3] R. Dijkgraaf and C. Vafa, *Matrix models, topological strings and supersymmetric gauge theories*, *Nucl. Phys.* **B 644** (2002) 3 [hep-th/0206255]; *On geometry and matrix models*, *Nucl. Phys.* **B 644** (2002) 21 [hep-th/0207106]; *A perturbative window into non-perturbative physics*, hep-th/0208048.

[4] A. Iqbal, *All genus topological string amplitudes and 5-brane webs as Feynman diagrams*, hep-th/0207114.

[5] D.-E. Diaconescu, B. Florea and A. Grassi, *Geometric transitions and open string instantons*, *Adv. Theor. Math. Phys.* **6** (2003) 619 [hep-th/0205234]; *Geometric transitions, del Pezzo surfaces and open string instantons*, *Adv. Theor. Math. Phys.* **6** (2003) 643 [hep-th/0206163].

[6] M. Aganagic, M. Marino and C. Vafa, *All loop topological string amplitudes from Chern-Simons theory*, *Commun. Math. Phys.* **247** (2004) 467 [hep-th/0206164].

[7] M. Aganagic, A. Klemm, M. Marino and C. Vafa, *The topological vertex*, *Commun. Math. Phys.* **254** (2005) 425 [hep-th/0305132].

[8] J. Li, Chiu-Chu Melissa Liu, K. Liu and J. Zhou, *A mathematical theory of the topological vertex*, math.AG/0408426.

[9] A. Iqbal and A.-K. Kashani-Poor, *Instanton counting and Chern-Simons theory*, *Adv. Theor. Math. Phys.* **7** (2004) 457 [hep-th/0212279].

[10] A. Iqbal and A.-K. Kashani-Poor, SU(N) *geometries and topological string amplitudes*, hep-th/0306032.

[11] T. Eguchi and H. Kanno, *Topological strings and Nekrasov's formulas*, *JHEP* **12** (2003) 006 [hep-th/0310235].

[12] T. Eguchi and H. Kanno, *Geometric transitions, Chern-Simons gauge theory and Veneziano type amplitudes*, *Phys. Lett.* **B 585** (2004) 163 [hep-th/0312234].

[13] J. Zhou, *Curve counting and instanton counting*, math.AG/0311237.

[14] A. Okounkov, N. Reshetikhin and C. Vafa, *Quantum Calabi-Yau and classical crystals*, hep-th/0309208.

[15] A. Iqbal, N. Nekrasov, A. Okounkov and C. Vafa, *Quantum foam and topological strings*, hep-th/0312022.

[16] D. Maulik, N. Nekrasov, A. Okounkov and R. Pandharipande, *Gromov-Witten theory and Donaldson-Thomas theory*, math.AG/0312059.

[17] D. Maulik, N. Nekrasov, A. Okounkov and R. Pandharipande, *Gromov-Witten theory and Donaldson-Thomas theory II*, math.AG/0406092.

[18] R. Dijkgraaf, S. Gukov, A. Neitzke and C. Vafa, *Topological M-theory as unification of form theories of gravity*, hep-th/0411073.

[19] N. Nekrasov, *A la recherche de la m-theorie perdue. Z theory: casing m/f theory*, hep-th/0412021.

[20] N. Nekrasov, H. Ooguri and C. Vafa, *S-duality and topological strings*, *JHEP* **10** (2004) 009 [hep-th/0403167].

[21] A. Kapustin, *Gauge theory, topological strings and S-duality*, *JHEP* **09** (2004) 034 [hep-th/0404041].

[22] N. Dorey, T.J. Hollowood, V.V. Khoze and M.P. Mattis, *The calculus of many instantons*, *Phys. Rept.* **371** (2002) 231 [hep-th/0206063].

[23] D. Bellisai, F. Fucito, A. Tanzini and G. Travaglini, *Instanton calculus, topological field theories and $N = 2$ super Yang-Mills theories*, *JHEP* **07** (2000) 017 [hep-th/0003272]; F. Fucito, J.F. Morales and A. Tanzini, *D-instanton probes of non-conformal geometries*, *JHEP* **07** (2001) 012 [hep-th/0106061].

[24] T.J. Hollowood, *Calculating the prepotential by localization on the moduli space of instantons*, *JHEP* **03** (2002) 038 [hep-th/0201075]; *Testing Seiberg-Witten theory to all orders in the instanton expansion*, *Nucl. Phys.* **B 639** (2002) 66 [hep-th/0202197].

[25] N.A. Nekrasov, *Seiberg-Witten prepotential from instanton counting*, *Adv. Theor. Math. Phys.* **7** (2004) 831 [hep-th/0206161].

[26] N. Nekrasov and A. Okounkov, *Seiberg-Witten theory and random partitions*, hep-th/0306238.

[27] H. Nakajima and K. Yoshioka, *Instanton Counting on Blowup I*, math.AG/0306198.

[28] H. Nakajima and K. Yoshioka, *Lectures on instanton counting*, math.AG/0311058.

[29] H.R Morton and S.G Lukac, *The HOMFLY polynomial of the decorated Hopf link*, math.GT/0108011.

[30] I.G. Macdonald, *Symmetric functions and Hall polynomials*, , Oxford University Press, Oxford 1995.

[31] R. Gopakumar and C. Vafa, *M-theory and topological strings, I*, hep-th/9809187.

[32] R. Gopakumar and C. Vafa, *M-theory and topological strings, II*, hep-th/9812127.

[33] T.J. Hollowood, A. Iqbal and C. Vafa, *Matrix models, geometric engineering and elliptic genera*, hep-th/0310272.

[34] S. Katz, A. Klemm and C. Vafa, *M-theory, topological strings and spinning black holes*, *Adv. Theor. Math. Phys.* **3** (1999) 1445 [hep-th/9910181].

[35] S. Hosono, M.H. Saito and A. Takahashi, *Holomorphic anomaly equation and BPS state counting of rational elliptic surface*, *Adv. Theor. Math. Phys.* **3** (1999) 177 [hep-th/9901151]; *Relative Lefschetz action and BPS state counting*, *Internat. Math. Res. Notices* **15** (2001) 783–816 [math.AG/0105148].

[36] S. Gukov, A. Schwarz and C. Vafa, *Khovanov-Rozansky homology and topological strings*, hep-th/0412243.

[37] H. Awata and H. Kanno, in preparation.

[38] Y. Tachikawa, *Five-dimensional Chern-Simons terms and Nekrasov's instanton counting*, *JHEP* **02** (2004) 050 [hep-th/0401184].

[39] P. Peng, *A simple proof of Gopakumar-Vafa conjecture for local toric Calabi-Yau manifolds*, math.AG/0410540.

[40] Y. Konishi, *Pole structure and Gopakumar-Vafa conjecture*, math.AG/0411357.

[41] A. Schwarz and I. Shapiro, *Some remarks on Gopakumar-Vafa invariants*, hep-th/0412119.

[42] R. Flume and R. Poghossian, *An algorithm for the microscopic evaluation of the coefficients of the Seiberg-Witten prepotential*, *Int. J. Mod. Phys.* **A 18** (2003) 2541 [hep-th/0208176].

[43] U. Bruzzo, F. Fucito, J.F. Morales and A. Tanzini, *Multi-instanton calculus and equivariant cohomology*, *JHEP* **05** (2003) 054 [hep-th/0211108].

[44] J. Li, K. Liu and J. Zhou, *Topological string partition functions as equivariant indices*, math.AG/0412089.